Introduction to Intuitionistic Logic

A proposal of a seminar for Erasmus+ students

Dr. Marcin Łazarz
Department of Logic and Methodology of Sciences
University of Wrocław

The topic of the seminar focuses on the one of the most important non-classical logics—intuitionistic logic. The philosophical background of this logic is so-called constructivism—a direction in the philosophy of mathematics developed by a Dutch mathematician Jan Brouwer in the twenties of the twentieth century. The fundamental Brouwer's idea is that it is necessary to point out ("to construct") a mathematical object to prove that it exists. In other words, if we show that the statement "the object x does not exist" is false, we then cannnot conclude that the statement "x exists" is true.

The program of constructivism is actually very restrictive and incompatible with many mathematical theories, especially the abstract ones, such as set theory, topology, algebra etc. As an example consider the following statement:

There exist irrational numbers x, y such that x^y is rational.

Proof. If $\sqrt{2}^{\sqrt{2}}$ is rational, we take $x = y = \sqrt{2}$; otherwise, by the law of excluded middle, $\sqrt{2}^{\sqrt{2}}$ is irrational, therefore we take $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$ and the proof is done. □

Although, the preceding argument seems to be clear, constructivists claim that it is basically incorrect, since it has not been decided whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational, and hence x and y are not properly constructed. Furthermore, Brouwer argues that, such non-effective proofs provide various paradoxes in the foundations of mathematics.

So what is the reason that proofs happen to be nonconstructive? According to constructivism, the answer is: because they use the laws of classical logic such as

$$\big(\neg p \to (q \wedge \neg q)\big) \to p, \quad p \vee \neg p, \quad \neg\neg p \to p$$

(so-called the law of *reductio ad absurdum*, the law of excluded middle, the law of double negation, respectively), etc. which are somehow "improper". Therefore, the principles of constructivism cannot be formalized on the ground of classical logic. Another Dutch logician Arend Heyting discovered in 1931 a logical calculus—now called intuitionistic logic—which is an adequate logical basis for constructivism.

On the seminar we discuss basic assumptions of the philosophy of constructivism, and present the syntax and the semantics of intuitionistic propositional logic. The course's main target is to provide the students with the theoretical notion of an intuitionistic tautology, and practical methods of a modern logical semantics helpful for deciding whether a given formula is an intuitionistic tautology.

Main topics:

1) Classical logic versus the philosophy of Jan Brouwer.
2) Non-constructive proofs.
3) The language of intuitionistic logic and the system INT.
4) The notion of formal proof in INT, the notion of a consistent set of sentences, the notion of a complete set of sentences.
5) Deduction theorem for INT.
6) Kripke semantics.
7) The notion of a tautology of INT.
8) The canonnical model and the completeness theorem for INT.
9) Intuitionistic logic versus classical logic: Gödel translation.
10) Gödel's theorem on the lack of a finite adequate matrix for INT.
11) Intermediate logics, disjunction property, Hallden property.

Recommended literature:

[1] D. Bridges, E. Palmgren, *Constructive Mathematics* in: **Stanford Encyclopedia of Philosophy**, 2013, http://plato.stanford.edu/entries/mathematics-constructive/

[2] R. Epstein, *The Semantic Foundations of Logic, Vol. 1: Propositional Logics*, Springer (Nijhoff International Philosophy Series 35), 1990.

[3] J. Moschovakis, *Intuitionistic Logic*, in: **Stanford Encyclopedia of Philosophy**, 2015, http://plato.stanford.edu/entries/logic-intuitionistic/